Short Course on Wolfram Mathematica

Lecture #2

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Sample Space S **is the set of all possible outcomes of an experiment.**

- **Example: the outcome of an experiment consists in the determination of the sex of a newborn child, then $S = \{g, b\}$, where the outcome g means that the child is a girl and b that it is a boy.**

Event E **is a subset of the *sample space* S. If the outcome of an experiment is contained in E, then we say that E has happened.**

- **Example: if $E = \{g\}$, then E is the event that the child is a girl. Similarly, if $F = \{b\}$, then F is the event that the child is a boy.**

Complement **event E^c consists of all outcomes in the sample space S that are not in E. That is, E^c will occur if E does not occur.**

- **Example: if $E = \{b\}$ is the event that the child is a boy, then $E^c = \{g\}$ is the event that it is a girl. Also, since the experiment must result in some outcome, it follows that $S^c = \emptyset$.**

Union $E \cup F$ **is an event consisted all outcomes that are either in event E or inevent F or in both.**

- **Example: if $E = \{g\}$ and $F = \{b\}$, then $E \cup F = \{g, b\}$ than $E \cup F$ is the whole sample space S.**

AXIOMS OF PROBABILITY

Probability **is a number between 0 and 1, where 0 indicates impossibility and 1 indicates certainty. The higher the probability of an event, the more likely it is that the event will occur.**

- **Example: the tossing of a coin: The two outcomes ("heads" and "tails") are both equally probable, then the probability of "heads" equals the probability of "tails“. Since no other outcomes are possible, the probability of either "heads" or "tails" is 0.5 (or 50%).**

If an experiment is continually repeated under the exact same conditions, then for any event E of sample space S, the proportion of time that the outcome is contained in E, approaches some constant. As the number of repetitions increases, this limiting constant of *relative frequency* is described as the *probability* $P(E)$ of the event E.

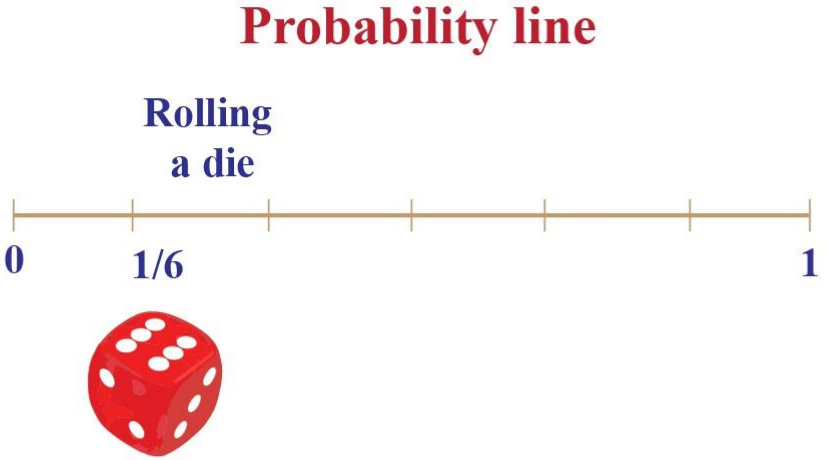
Axiom I	$0 \le P(E) \le 1$
Axiom II	$P(S) = 1$
Axiom III	$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$, **where** $n = 1, 2, \ldots, \infty$ **and** $E_i E_j = 0$ **if** $i \neq j$ **(events** E_i **and** E_j **are mutually exclusive).**

Example: someone is tossing a coin ($n = 1$ and $N = 2$), then $P(E_{head}) = P(E_{tail}) = \frac{1}{2}$ *and* $p = \frac{1}{2}$.

Example: Find $P(S_i) = p$, where S_i is the outcome when we roll one fair die.

Solution: $n = 1, N = 6 \implies$

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = p = \frac{n}{N} = \frac{1}{6}\ .$$



Two Propositions of Probabilities

Events E and E^c are always mutually exclusive, and since $E \cup E^c = S$, by using Axioms II and III one can get PROPOSITION #1:

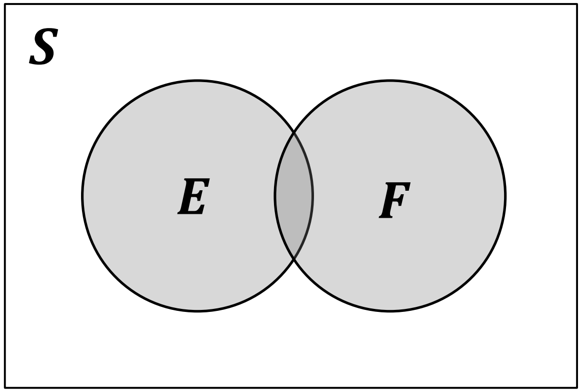
$$P(S) = 1 = P(E \cup E^c) = P(E) + P(E^c) \implies \quad P(E^c) = 1 - P(E).$$

PROPOSITION #2 gives the relationship between the probability of the union $P(E \cup F)$ of events E and F in terms of the individual probabilities $P(E)$ and $P(F)$ and the probability of the intersection $P(EF)$:

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

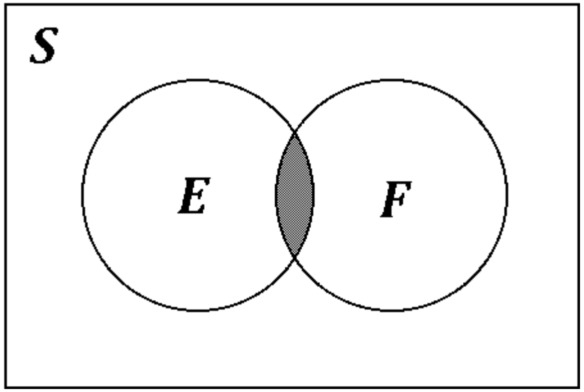
Union

Shaded region: $E \cup F$



Intersection

Shaded region: EF



EXAMPLE. A total of 28 percent of American males smoke cigarettes, 7 percent smoke cigars, and 5 percent smoke both cigars and cigarettes. What percentage of males smoke neither cigars nor cigarettes?

SOLUTION: Let's assume that E is the event that a randomly chosen male is a cigarette smoker and event F is the event that he is a cigar smoker. Then, the probability this person is a smoker is

$$P(E \cup F) = P(E) + P(F) - P(EF) = 0.28 + 0.07 - 0.05 = 0.3$$

By using Axiom II, one can show that the probability that the person is not a smoker is

$$1 - P(E \cup F) = 1 - 0.3 = 0.7\ ,$$

implying that 70 percent of American males smoke neither cigarettes nor cigars.

- **Definition: *Probability mass function* $p(a)$ of a discrete random variable X is defined by the equation:**

$$\boldsymbol{p(a) = P\{X = a\}.}$$

Properties of *probability mass function* for its possible values of $S = \{x_i, \dots, x_\infty\}$

$p(x_i) > 0$ and $\sum_{i=1}^{\infty} p(x_i) = P\{\bigcup_{i=1}^{\infty} x_i\} = P\{S\} = 1$, where $i = 1, 2, 3, \dots \infty$ **and** $x_\infty \neq \infty$.

For other values of x outside of sample space S the random variables set $p(x) = 0$.

The *cumulative distribution function* F can be expressed in terms of $p(x)$:

$$F(a) = P\{X \leq a\} = \sum_{all\ x \leq a} p(x)$$

Expectation

- ***Definition*: *Expectation* of a discrete random variable X is a weighted average of the possible values x_i that X can take on, where each value being weighted by the probability $P\{X = x_i\}$:**

$$E[X] = \sum_{i}^{n} x_i P\{X = x_i\} = \sum_{i}^{n} x_i p(x_i) = \mu.$$

EXAMPLE. Find $E[X]$ where X is the outcome when we roll one fair die.

Solution: $p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = \frac{1}{6}$.

$$E[X] = \sum_i^n x_i p(x_i) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) =$$

$$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2} = 3.5\,.$$

Conclusion:

- ***Expectation* $E[X] = \mu$ is the *true mean* value of all random variables belonging to the population of X.**
- **The sample mean $\bar{X}$ is not necessarily equal to true mean μ.**

The Bernoulli and Binomial Random Variables

For some experiments one can consider only two outcomes, namely, "success" or "failure". Let's assume $X = 1$ when the outcome is a success and $X = 0$ when it is a failure.

- **Definition: Bernoulli random variable is a random variable X with its probability mass function is given by $P\{X = 1\} = p$ and $P\{X = 0\} = 1 - p$, where $0 \le p \le 1$. Both probabilities are described for a single trial $(n = 1)$.**

The expected value $E[X]$ is governed by

$$E[X] = 1 \cdot P\{X = 1\} + 0 \cdot P\{X = 0\} = p.$$

Example: It can be used to represent a coin toss where 1 and 0 would represent "heads" and "tails", respectively. Thus, the probability mass function of the coin landing on heads or tails, respectively, is $P\{X = 1\} = P\{X = 0\} = p = \frac{1}{2}$

- **Definition: The *binomial* random variable X with parameters (n, p) represent the number of "successes" that occur in the $n > 1$ independent Bernoulli trials with the probability p (and "failure" with probability $1 - p$): $X = \sum_{i=1}^{n} X_i$.**

The probability mass function of a *binomial* random variable is given by

$$P\{X = i\} = \binom{n}{i} p^i (1 - p)^{n-i}, \qquad i = 0, 1, \ldots, n$$

Where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is the number of different groups of i objects that can be chosen from a set of n objects, where the n outcomes contain i successes and $(n - i)$ failures.

Note that one can troubleshoot those formulas as next

$$P\{S\} = \sum_{i=0}^{n} \binom{n}{i} p^i (1 - p)^{n-i} = \left| \begin{matrix} Bernoulli \\ \\ n = 1 \end{matrix} \right| = \binom{1}{0} p^0 (1 - p)^1 + \binom{1}{1} p^1 (1 - p)^0 = (1 - p) + p = 1$$

EXAMPLE. **A communications system consists of n components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one-half of its components function.**

For what values of p is a 5-component system more likely to operate effectively than a 3-component system?

Solution: The probability that a 5-component system will be effective is

$$P(3 \le X \le 5) = \sum_{i}^{n} p(i) = \sum_{i}^{n} \binom{n}{i} p^{i}(1-p)^{n-i} = \left| \begin{matrix} i = 3 \\ \\ n = 5 \end{matrix} \right| = \sum_{i=3}^{5} \binom{5}{i} p^{i}(1-p)^{5-i} =$$

$$= \binom{5}{3} p^{3}(1-p)^{5-3} + \binom{5}{4} p^{4}(1-p)^{5-4} + \binom{5}{5} p^{5}(1-p)^{5-5} =$$

$$= 10p^{3}(1-p)^{2} + 5p^{4}(1-p) + p^{5};$$

The probability that a 3-component system will be effective is

$$P(2 \le X \le 3) = \sum_{i=2}^{3} \binom{3}{i} p^{i}(1-p)^{3-i} =$$

$$= \binom{3}{2} p^{2}(1-p) \quad + \binom{3}{3} p^{3}(1-p)^{0} = 3p^{2}(1-p) \quad + p^{3}.$$

EXAMPLE. A communications system consists of n components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one-half of its components function.

For what values of p is a 5-component system more likely to operate effectively than a 3-component system?

Solution: The probability that a 5-component system will be effective is

$$\sum_{i=3}^{5}\binom{5}{i}p^i(1-p)^{5-i} = 10p^3(1-p)^2 + 5\mathrm{p}(1-p) + p^5;$$

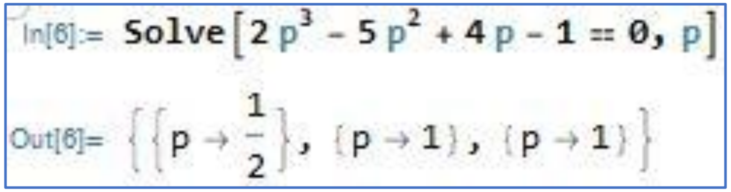
The probability that a 3-component system will be effective is

$$\sum_{i=2}^{3}\binom{3}{i}p^i(1-p)^{3-i} = 3p^2(1-p) \quad + p^3.$$

Thus, the 5-component system is better if

$$10p^3(1-p)^2 + 5p^4(1-p) + p^5 > 3p^2(1-p) \quad + p^3$$

$$2p^3 - 5p^2 + 4p - 1 > 0 \;\Longrightarrow\; \text{Answer:}\, p > \frac{1}{2}$$



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In[6]:= Solve[2 p^3 - 5 p^2 + 4 p - 1 == 0, p]

Out[6]= {{p → 1/2}, {p → 1}, {p → 1}}
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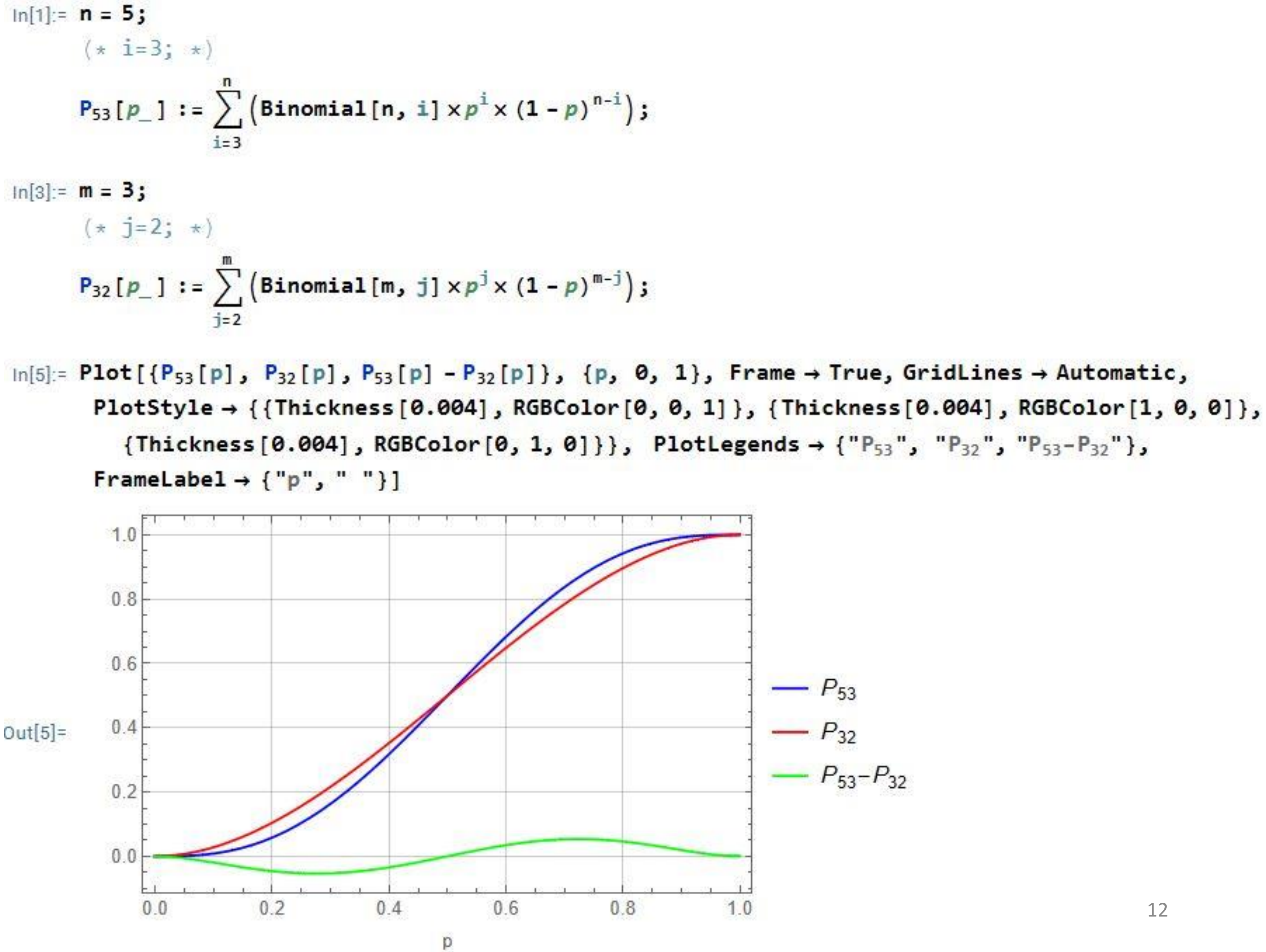


```
In[1]:= n = 5;
        (* i=3; *)
```

$$P_{53}[p\_] := \sum_{i=3}^{n}\left(\mathrm{Binomial}[n, i] \times p^{i} \times (1-p)^{n-i}\right);$$

```
In[3]:= m = 3;
        (* j=2; *)
```

$$P_{32}[p\_] := \sum_{j=2}^{m}\left(\mathrm{Binomial}[m, j] \times p^{j} \times (1-p)^{m-j}\right);$$

```
In[5]:= Plot[{P53[p], P32[p], P53[p] - P32[p]}, {p, 0, 1}, Frame → True, GridLines → Automatic,
         PlotStyle → {{Thickness[0.004], RGBColor[0, 0, 1]}, {Thickness[0.004], RGBColor[1, 0, 0]},
           {Thickness[0.004], RGBColor[0, 1, 0]}}, PlotLegends → {"P53", "P32", "P53-P32"},
         FrameLabel → {"p", " "}]
```

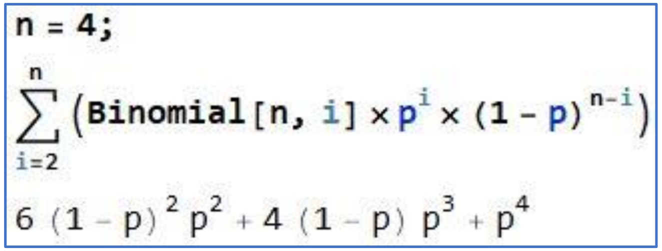
Out[5]=

Problem. At least one-half of an airplane's engines are required to function in order for it to operate. If each engine independently functions with probability p, for what values of p is a 4-engine plane more likely to operate than a 2-engine plane?

Solution:

$$P(2 \le X \le 4) = \sum_{i}^{n} p(i) = \left| \begin{matrix} n = 4 \\ \\ i = 2 \end{matrix} \right| = \sum_{i=2}^{4} \binom{4}{2} p^{i}(1-p)^{n-i} =$$

$$= 6(1-p)^2 p^2 + 4(1-p)p^3 + p^4$$

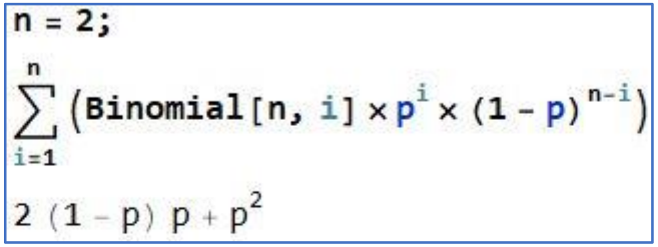


$$\texttt{n = 4;}$$

$$\sum_{i=2}^{n} \left(\texttt{Binomial}[n, i] \times p^{i} \times (1-p)^{n-i}\right)$$

$$6\,(1-p)^2\,p^2 + 4\,(1-p)\,p^3 + p^4$$

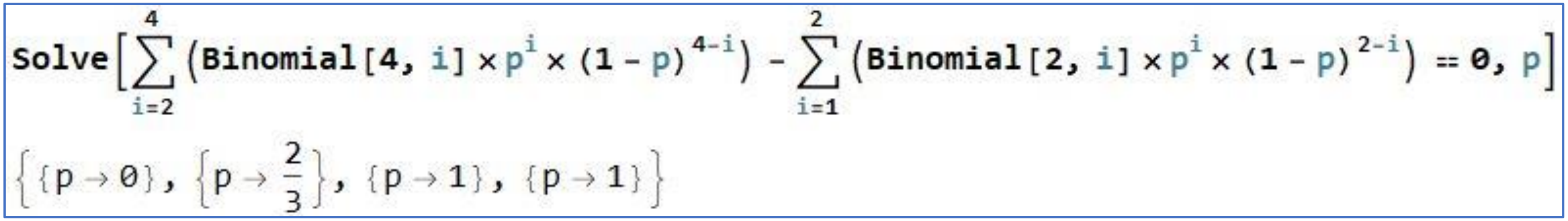
$$P(1 \le X \le 2) = \sum_{i}^{n} p(i) = \left| \begin{matrix} n = 2 \\ \\ i = 1 \end{matrix} \right| = \sum_{i=1}^{2} \binom{2}{1} p^{i}(1-p)^{n-i} = 2(1-p)p + p^2$$



$$\texttt{n = 2;}$$

$$\sum_{i=1}^{n} \left(\texttt{Binomial}[n, i] \times p^{i} \times (1-p)^{n-i}\right)$$

$$2\,(1-p)\,p + p^2$$



$$\texttt{Solve}\left[\sum_{i=2}^{4} \left(\texttt{Binomial}[4, i] \times p^{i} \times (1-p)^{4-i}\right) - \sum_{i=1}^{2} \left(\texttt{Binomial}[2, i] \times p^{i} \times (1-p)^{2-i}\right) == 0,\ p\right]$$

$$\left\{\{p \to 0\},\ \left\{p \to \frac{2}{3}\right\},\ \{p \to 1\},\ \{p \to 1\}\right\}$$

Answer: $p > \frac{2}{3} = 0.6(6)$

In[1]:= n = 4;

(* i=2; *)

$$P_{42}[p\_] := \sum_{i=2}^{n} \left(\text{Binomial}[n, i] \times p^{i} \times (1-p)^{n-i}\right);$$

In[3]:= m = 2;

(* j=1; *)

$$P_{21}[p\_] := \sum_{j=1}^{m} \left(\text{Binomial}[m, j] \times p^{j} \times (1-p)^{m-j}\right);$$

```
In[5]:= Plot[{P42[p], P21[p], P42[p] - P21[p]}, {p, 0, 1}, Frame → True, GridLines → Automatic,
         PlotStyle → {{Thickness[0.004], RGBColor[0, 0, 1]}, {Thickness[0.004], RGBColor[1, 0, 0]},
           {Thickness[0.004], RGBColor[0, 1, 0]}},  PlotLegends → {"P42", "P21", "P42-P21"},
         FrameLabel → {"p", " "}]
```

Out[5]=